DS-GA 1014 - Homework 1

Eric Niblock

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1. Are the following sets subspaces of $\mathbb{R}^3$? Justify your answer.

(a) $E_1 = \{(x, y, z) \in \mathbb{R}^3 | \ x - 2y + z = 0\}$

In order to show that E_1 is a subspace of $\mathbb{R}^3$, we must show that E_1 contains the zero-vector, is closed under vector-addition, and is closed under scalar multiplication. It is easy enough to verify that the zero-vector, $(x, y, z) = (0, 0, 0)$ is contained since,

$$(0) - 2(0) + (0) = 0 \tag{1}$$

In order to show the other two properties, imagine that we have two vectors which lie within E_1, and are given by $\overrightarrow{\mathbf{u}} = (x_1, y_1, z_1)$ and $\overrightarrow{\mathbf{v}} = (x_2, y_2, z_2)$. Then, this implies that,

$$(x_1) - 2(y_1) + (z_1) = 0 \tag{2}$$
$$(x_2) - 2(y_2) + (z_2) = 0 \tag{3}$$

Furthermore, the sum of these two equations is also implied to be true, so we have,

$$(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2) = 0 \tag{4}$$

Which suggests that some other vector, $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$, is also within E_1. However, this new vector is the vector $\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}$. So, we have shown that $\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}} \in E_1$ implies $\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} \in E_1$, and that E_1 is closed under vector addition.

Lastly, we can multiply (2) by some scalar constant c, to obtain,

$$c((x_1) - 2(y_1) + (z_1)) = c(0) \tag{5}$$
$$(cx_1) - 2(cy_1) + (cz_1) = 0 \tag{6}$$

Which implies that some some other vector, (cx_1, cy_1, cz_1), is also within E_1. However, this new vector is the vector $c\vec{\mathbf{u}}$. So, we have shown that $\vec{\mathbf{u}} \in E_1$ implies $c\vec{\mathbf{u}} \in E_1$, and that E_1 is closed under scalar multiplication.

Because E_1 has satisfied these three conditions, E_1 is a subspace of $\mathbb{R}^3$.

(b) $E_2 = \{(x, y, z) \in \mathbb{R}^3 | \; x - 2y + z = 3\}$

It is clear that E_2 is not a subspace of $\mathbb{R}^3$, since the zero-vector, $(x, y, z) = (0, 0, 0)$, is not contained,

$$(0) - 2(0) + (0) \neq 3 \tag{7}$$

(c) $E_3 = \{(x, y, z) \in \mathbb{R}^3 | \; 5x - y^2 + z = 0\}$

It is clear that E_3 is not a subspace of $\mathbb{R}^3$ because it does not satisfy the property of being closed under vector-addition. Take the following vectors, which both lie in the subspace: $\vec{\mathbf{u}}$ = (0,1,1), $\vec{\mathbf{v}}$ = (0,2,4). Now, we can see that $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ is not a solution,

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (0, 3, 5) \tag{8}$$
$$5(0) - (3)^2 + 5 \neq 0 \tag{9}$$

2. Let $\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}} \in \mathbb{R}^n$. Assume that $\overrightarrow{\mathbf{x_1}} \in Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}})$. Show that

$$Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}}) = Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}}) \tag{10}$$

Observe the following definition of linear span,

> *The linear span of vectors $\overrightarrow{\boldsymbol{x_1}}, ..., \overrightarrow{\boldsymbol{x_k}}$ is the set of all linear combinations of these vectors. [Def. 1]*

Given this definition, we can write the following spans as follows

$$Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}}) = \{a_1\overrightarrow{\mathbf{x_1}} + ... + a_k\overrightarrow{\mathbf{x_k}} \,|\, a_i \in \mathbb{R}\} \tag{11}$$
$$Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}}) = \{b_2\overrightarrow{\mathbf{x_2}} + ... + b_k\overrightarrow{\mathbf{x_k}} \,|\, b_i \in \mathbb{R}\} \tag{12}$$

It then becomes clear that $Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}}) \subseteq Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}})$ because when $a_1 = 0$, the constants a_i, b_i can be matched to produce the same set of vectors. Since $\overrightarrow{\mathbf{x_1}} \in Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}})$, we can write

$$\overrightarrow{\mathbf{x_1}} = \sum_{i=2}^{k} b_i^{'}\overrightarrow{\mathbf{x_i}} \tag{13}$$

With $b_i^{'} \in \mathbb{R}$ for all i. This then allows us to write,

$$Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}}) = \{(a_1b_2^{'} + a_2)\overrightarrow{\mathbf{x_2}} + ... + (a_1b_k^{'} + a_k)\overrightarrow{\mathbf{x_k}} \,|\, a_i \in \mathbb{R}\} \tag{14}$$
$$Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}}) = \{b_2\overrightarrow{\mathbf{x_2}} + ... + b_k\overrightarrow{\mathbf{x_k}} \,|\, b_i \in \mathbb{R}\} \tag{15}$$

This implies that $Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}}) \subseteq Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}})$ since the constants $a_1b_i^{'}+a_i$,b_i can always be matched to produce the same set of vectors. Therefore, since we have shown $Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}}) \subseteq Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}})$ and $Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}}) \subseteq Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}})$, we have,

$$Span(\overrightarrow{\mathbf{x_1}}, ..., \overrightarrow{\mathbf{x_k}}) = Span(\overrightarrow{\mathbf{x_2}}, ..., \overrightarrow{\mathbf{x_k}}) \tag{16}$$

3. Suppose that $\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k} \in \mathbb{R}^n$ are linearly independent. Let $\overrightarrow{\mathbf{x}} \in \mathbb{R}^n$ and assume that $\overrightarrow{\mathbf{x}} \notin Span(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$. Show that $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k}, \overrightarrow{\mathbf{x}})$ are linearly independent.

Proof by contradiction. Assume that $\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k} \in \mathbb{R}^n$ are linearly independent, $\overrightarrow{\mathbf{x}} \in \mathbb{R}^n$, $\overrightarrow{\mathbf{x}} \notin Span(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$, and $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k}, \overrightarrow{\mathbf{x}})$ are not a group of linearly independent vectors.

From this, we can assume that the appending of $\overrightarrow{\mathbf{x}}$ to $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$ must have collapsed the property of linear independence because $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$ is independent but $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k}, \overrightarrow{\mathbf{x}})$ is not. Therefore, $\overrightarrow{\mathbf{x}}$ must be a linear combination of the vectors $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$,

$$\overrightarrow{\mathbf{x}} = \sum_{i=1}^{k} c_i \overrightarrow{\mathbf{v}_i} \tag{17}$$

With $c_i \in \mathbb{R}$ for all i. This is because of the definition of linear dependency,

> *Vectors $\overrightarrow{\boldsymbol{x}_1}, ..., \overrightarrow{\boldsymbol{x}_k}$ are linearly dependent if one of them can be expressed as a linear combination of the others. Otherwise, these vectors are said to be linearly independent. [Def. 2]*

Again, employing *Definition 1* from above, we see that if $\overrightarrow{\mathbf{x}} \notin Span(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$, then it must follow that $\overrightarrow{\mathbf{x}}$ is not a linear combination of the vectors $(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_k})$. In other words,

$$\overrightarrow{\mathbf{x}} \neq \sum_{i=1}^{k} c_i \overrightarrow{\mathbf{v}_i} \tag{18}$$

With $c_i \in \mathbb{R}$ for all i. Contradiction. Therefore, we have shown the original statement of the problem.

4. We prove in this problem Proposition 3.2 from the notes. You can use the results of Problems 1.2 and 1.3 and of course the other results from the lecture.

Let S be a subspace of $\mathbb{R}^n$ of dimension k and let $\overrightarrow{x_1}, ..., \overrightarrow{x_k} \in S$.

(a) Show that if $\overrightarrow{x_1}, ..., \overrightarrow{x_k}$ are linearly independent, then $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ is a basis of S

Proof by contradiction. Suppose $(\overrightarrow{x_1}, ..., \overrightarrow{x_k}) \in S$ is not a basis of S, a subspace with dimension k, and that $\overrightarrow{x_1}, ..., \overrightarrow{x_k}$ are linearly independent.

Observe the following definition of basis,

> *A family $(\overrightarrow{x_1}, ..., \overrightarrow{x_n})$ of vectors in V form a basis if (1) $(\overrightarrow{x_1}, ..., \overrightarrow{x_n})$ are linearly independent and (2) Span$(\overrightarrow{x_1}, ..., \overrightarrow{x_n}) = V$. [Def. 3]*

Given that, from our assumption, (1) is satisfied, it must be that $Span(\overrightarrow{x_1}, ..., \overrightarrow{x_k}) \neq S$. However, this would imply that there exists, at a minimum, some $\overrightarrow{v} \in S$ such that $\overrightarrow{v} \notin Span(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$. By *Problem 3*, this implies that $(\overrightarrow{x_1}, ..., \overrightarrow{x_k}, \overrightarrow{v})$ are linearly independent. However, observe the following proposition,

> *Let V be a vector space that has dimension $dim(V) = n$. Then any family of vectors of V that are linearly independent contains at most n vectors [Prop. 1]*

Proposition 1 implies that $(\overrightarrow{x_1}, ..., \overrightarrow{x_k}, \overrightarrow{v})$ is not linearly independent, because this family contains $k + 1$ vectors. Contradiction.

Therefore, it has been shown that if $\overrightarrow{x_1}, ..., \overrightarrow{x_k}$ are linearly independent, then $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ is a basis of S.

(b) Show that if $Span(\overrightarrow{x_1}, ..., \overrightarrow{x_k}) = S$, then $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ is a basis of S.

Proof by contradiction. Assume that $Span(\overrightarrow{x_1}, ..., \overrightarrow{x_k}) = S$, S has dimension k, and that $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ is not a basis of S.

According to *Definition 3*, $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ is a basis of S if two conditions are satisfied. We know that $Span(\overrightarrow{x_1}, ..., \overrightarrow{x_k}) = S$, so the second condition is satisfied. Therefore, $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ must not be linearly independent. Since we have assumed that $(\overrightarrow{x_1}, ..., \overrightarrow{x_k})$ are linearly dependent, there exists some $i \in \{1, ..., k\}$ such that $x_i \in Span((x_j)_{j \neq i})$. One can assume that $i = k$ by permuting the rows, and hence, by *Problem 2*,

$$Span(\overrightarrow{x_1}, ..., \overrightarrow{x_k}) = Span(\overrightarrow{x_1}, ..., \overrightarrow{x_{k-1}}) = S \tag{19}$$

However, observe

> *Let V be a vector space that has dimension $dim(V) = n$. Then any family of vectors of V that spans V contains at least n vectors [Prop. 2]*

This implies that

$$Span(\overrightarrow{\mathbf{x}_1}, ..., \overrightarrow{\mathbf{x}_{k-1}}) \neq S \tag{20}$$

Contradiction. The original statement has been proven.