DS-GA 1014 - Homework 11

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1. (2 points). Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be the functions defined by

$$f(x,y,z) = 2x^2 + y^2 + \frac{1}{2}z^2 + 4x - 6y - z + 1 \tag{1}$$
$$g(x,y,z) = -xyz + x + y + z \tag{2}$$

Compute the critical points of f and g and determine if they are global/local maximizers/minimizers or saddle points.

Concerning f, we have,

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x+4 \\ 2y-6 \\ z-1 \end{bmatrix} \tag{3}$$

Setting $\nabla f = 0$ implies that a critical point occurs at $(x, y, z) = (-1, 3, 1)$. Furthermore, the Hessian, H_f is given by,

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Since H_f is a diagonal matrix, the eigenvalues can be read directly from the diagonal. All of the eigenvalues are strictly positive implying that the critical point $(x, y, z) = (-1, 3, 1)$ is a local minimum. Upon further inspection of the gradient, we realize this is also a global minimum.

Concerning g, we have,

$$\nabla g = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} -yz+1 \\ -xz+1 \\ -xy+1 \end{bmatrix} \tag{5}$$

Setting $\nabla g = 0$ implies that a critical point occurs when $(x, y, z) = (1, 1, 1)$ and when $(x, y, z) = (-1, -1, -1)$. Furthermore, the Hessian, H_g is given by,

$$H_g = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 0 & -z & -y \\ -z & 0 & -x \\ -y & -x & 0 \end{bmatrix} \tag{6}$$

Then our critical points have the associated Hessians,

$$H_g(1,1,1) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \quad H_g(-1,-1,-1) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{7}$$

The critical point $H_g(1, 1, 1)$ has eigenvalues $\lambda_2 = -2$ and $\lambda_{1,3} = 1$, and is therefore a saddle point. The critical point $H_g(-1, -1, -1)$ has eigenvalues $\lambda_{1,2} = -1$ and $\lambda_3 = 2$, and is therefore a saddle point.

2. (3 points). We consider the following constrained optimization problem:

minimize $x - y + z$ subject to $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$

We admit that this minimization problem has (at least) one solution (this comes from the fact that a continuous function on a compact set attains its minimum). Using Lagrange multipliers, show that the minimization problem has a unique solution and compute its coordinates.

We have the following general formula for the Lagrangian concerning our problem,

$$\mathcal{L}_{\lambda_1,\lambda_2}(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1 g_1(x, y, z) + \lambda_2 g_2(x, y, z) \tag{8}$$

Where f represents the function that is to be minimized, and g_1 and g_2 represent the constraints. Then,

$$\mathcal{L}_{\lambda_1,\lambda_2}(x,y,z,\lambda_1,\lambda_2) = x - y + z + \lambda_1(x^2+y^2+z^2-1) + \lambda_2(x+y+z-1) \tag{9}$$

Then, $\nabla\mathcal{L}$ provides us with a set of equations to be solved simultaneously,

$$\frac{\partial \mathcal{L}}{\partial x} = 1 + 2\lambda_1 x + \lambda_2 = 0 \implies x = \frac{-1-\lambda_2}{2\lambda_1} \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -1 + 2\lambda_1 y + \lambda_2 = 0 \implies y = \frac{1-\lambda_2}{2\lambda_1} \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 1 + 2\lambda_1 z + \lambda_2 = 0 \implies z = \frac{-1-\lambda_2}{2\lambda_1} \implies x = z \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = x^2+y^2+z^2-1 = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = x+y+z-1 = 0 \tag{14}$$

Then we have,

$$x+y+z = 1 \implies 2\left(\frac{-1-\lambda_2}{2\lambda_1}\right) + \frac{1-\lambda_2}{2\lambda_1} = 1 \implies \lambda_1 = \frac{-3\lambda_2-1}{2} \tag{15}$$

$$x^2+y^2+z^2 = 1 \implies 2\left(\frac{-1-\lambda_2}{2\lambda_1}\right)^2 + \left(\frac{1-\lambda_2}{2\lambda_1}\right)^2 = 1 \implies \lambda_2 = \frac{-1\pm 2}{3} \tag{16}$$

And knowing the values of λ_1 and λ_2 implies that we have two sets of possible values in terms of coordinates: $(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$ and $(0,1,0)$. Both of these sets of coordinates satisfy the constraint equations, it simply remains to be seem which provides a lower functional value for f.

$$f(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}) = \frac{5}{3} \tag{17}$$

$$f(0,1,0) = -1 \tag{18}$$

Therefore $(x, y, z) = (0, 1, 0)$ corresponds to the minimum of f which satisfies the constraint equations.

3. (2 points). Let $\vec{\mathbf{u}} \in \mathbb{R}^n$ be a vector such that for all $i \neq j, |\vec{\mathbf{u}}_i| \neq |\vec{\mathbf{u}}_j|$. We consider the constrained optimization problem,

$$\textbf{maximize} \quad \langle \vec{\mathbf{u}}, \vec{\mathbf{x}} \rangle \quad \textbf{subject to} \quad ||\vec{\mathbf{x}}||_1 \leq 1$$

(a) Show that this problem has a unique solution $\vec{\mathbf{x}}^*$ and give the expression of $\vec{\mathbf{x}}^*$ in terms of $\vec{\mathbf{u}}$ (Lagrange multipliers are not needed here).

We first note that maximizing $\langle \vec{\mathbf{u}}, \vec{\mathbf{x}} \rangle$ is equivalent to maximizing $u_1x_1 + ... + u_nx_n$. Furthermore, we know that $||\vec{\mathbf{x}}||_1 \leq 1$ is equivalent to $|x_1| + ... + |x_n| \leq 1$. We then have that,

$$\begin{aligned} \langle \vec{\mathbf{u}}, \vec{\mathbf{x}} \rangle &= u_1x_1 + ... + u_nx_n \\ &\leq |u_1||x_1| + ... + |u_n||x_n| \end{aligned} \tag{19}$$

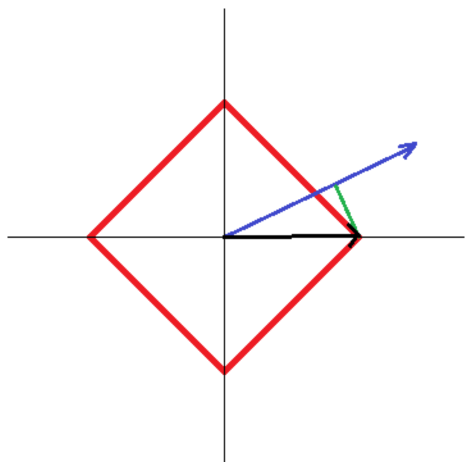
Then we define some i^* such that for all $j \in \{1, ..., n\}$ we have $|u_i^*| \geq |u_j|$. Then, we have that every for every u_j we can write the expression $|u_j| = |u_i^*| - \alpha_j$ such that $\alpha_j \geq 0$. Then, continuing from above, we have,

$$\begin{aligned} \langle \vec{\mathbf{u}}, \vec{\mathbf{x}} \rangle &= u_1x_1 + ... + u_nx_n \\ &\leq |u_1| \cdot |x_1| + ... + |u_n| \cdot |x_n| \\ &= |x_1| \cdot (|u_i^*| - \alpha_1) + ... + |x_n| \cdot (|u_i^*| - \alpha_n) \\ &= |u_i^*| \cdot ||\vec{\mathbf{x}}||_1 - \sum_{i^* \neq j}^{n} \alpha_j \cdot |x_j| \\ &\leq \langle \vec{\mathbf{u}}, \vec{\mathbf{x}}_i^* \rangle \end{aligned} \tag{20}$$

Thus this shows that the solution is given by $\vec{\mathbf{x}}^* = \vec{\mathbf{x}}_i^*$, that is a zector of all zeros except for the i^* location, which is populated with ± 1 (the same sign of u_i^*). Suppose is is not the case that $\vec{\mathbf{x}}^* = \vec{\mathbf{x}}_i^*$, but instead $\vec{\mathbf{x}}^* = \vec{\mathbf{x}}$. If $\vec{\mathbf{x}}^* = \vec{\mathbf{x}}$, then one possibility is that for some $i \neq i^*$, $x_i \neq 0$. If this is the case, then we see by the inequality that the expression is reduced by the corresponding term within the summation, and thus cannot produce the maximum $\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{x}}_i^* \rangle$. Thus, we have shown that $\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{x}} \rangle$ is maximized when $\overrightarrow{\mathbf{x}}^* = \overrightarrow{\mathbf{x}}_i^*$, as defined above.

(b) Give a graphical interpretation.

Observe the following graphic and explanation,



The graphic displays the canonical axes in $\mathbb{R}^2$. In red, the boundaries of the ℓ_1 norm are shown, and in blue a random vector $\overrightarrow{\mathbf{u}} \in \mathbb{R}^2$ has been chosen. Furthermore, here we have shown the optimal solution for our choice of $\overrightarrow{\mathbf{x}}$, given in black. Note that $\langle \overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{x}} \rangle$ would result in a vector lying along $\overrightarrow{\mathbf{u}}$, and ending where the green vector intersects $\overrightarrow{\mathbf{u}}$. It is clear that no other $\overrightarrow{\mathbf{x}}$ can be drawn such that it falls within or upon the boundary given in red, and fosters a larger vector on $\overrightarrow{\mathbf{u}}$. In fact, it is clear that if we rotate $\overrightarrow{\mathbf{u}}$ by 30 degrees counter-clockwise, $\overrightarrow{\mathbf{x}}$ would snap to the vertical canonical axis, in order to produce the largest inner product. A similar process occurs in higher dimensions, thus, the optimal $\overrightarrow{\mathbf{x}}$ always lies in the direction of one of the canonical axes, which corresponds to the component of $\overrightarrow{\mathbf{u}}$ that has the greatest magnitude.

4. (3 points). We will prove the spectral theorem in this problem: you are therefore not allowed to use the spectral theorem and its consequences to solve this exercise.

Let A be an $n \times n$ symmetric matrix. We consider the following optimization problem,

$$\textbf{maximize} \quad \overrightarrow{\mathbf{x}}^T A \overrightarrow{\mathbf{x}} \quad \textbf{subject to} \quad ||\overrightarrow{\mathbf{x}}|| = 1$$

This optimization problem admits a solution (this comes from the fact that a continuous function on a compact set achieved its maximum) that we denote by $\overrightarrow{\mathbf{v}}_1$.

(a) Using Lagrange multipliers, show that $\vec{\mathbf{v}}_1$ is an eigenvector of A.

We wish to minimize $-\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$ (which is equivalent to maximizing $\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$) under the constraint that $||\vec{\mathbf{x}}|| = 1$. By use of Lagrange multipliers, we have that,

$$\mathcal{L}_{\lambda_1}(\vec{\mathbf{x}}, \lambda_1) = -\vec{\mathbf{x}}^T A \vec{\mathbf{x}} + \lambda_1(\vec{\mathbf{x}}^T \vec{\mathbf{x}} - 1) \tag{21}$$

Then, we have that,

$$\frac{\partial \mathcal{L}_{\lambda_1}(\vec{\mathbf{x}}, \lambda_1)}{\partial \vec{\mathbf{x}}} = -2A\vec{\mathbf{x}} + 2\lambda_1 \vec{\mathbf{x}} = 0 \tag{22}$$

$$A\vec{\mathbf{x}} = \lambda_1 \vec{\mathbf{x}} \tag{23}$$

Which suffices to show that $(\lambda_1, \vec{\mathbf{x}})$ is an eigenvalue, eigenvector pair (since . We will refer to this pair as $(\mu_1, \vec{\mathbf{v}}_1)$ to avoid confusion in future parts.

(b) We now consider the optimization problem

maximize $\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$ **subject to** $||\vec{\mathbf{x}}|| = 1$ **and** $\langle \vec{\mathbf{x}}, \vec{\mathbf{v}}_1 \rangle = 0$

For the same reason as above, this problem admits a solution that we denote by $\vec{\mathbf{v}}_2$. Show that $\vec{\mathbf{v}}_2$ is an eigenvector of A that is orthogonal to $\vec{\mathbf{v}}_1$.

Again, we wish to minimize $-\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$ (which is equivalent to maximizing $\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$) under the constraints that $||\vec{\mathbf{x}}|| = 1$, and $\langle \vec{\mathbf{x}}, \vec{\mathbf{v}}_1 \rangle = 0$. Then, by use of Lagrange multipliers, we have,

$$\mathcal{L}_{\lambda_1,\lambda_2}(\vec{\mathbf{x}}, \vec{\mathbf{v}}_1, \lambda_1, \lambda_2) = -\vec{\mathbf{x}}^T A \vec{\mathbf{x}} + \lambda_1(\vec{\mathbf{x}}^T \vec{\mathbf{x}} - 1) + \lambda_2 \vec{\mathbf{x}}^T \vec{\mathbf{v}}_1 \tag{24}$$

Then, we have that,

$$\frac{\mathcal{L}_{\lambda_1,\lambda_2}(\vec{\mathbf{x}}, \vec{\mathbf{v}}_1, \lambda_1, \lambda_2)}{\partial \vec{\mathbf{x}}} = -2A\vec{\mathbf{x}} + 2\lambda_1 \vec{\mathbf{x}} + \lambda_2 \vec{\mathbf{v}}_1 = 0 \tag{25}$$

If we multiply by $\overrightarrow{\mathbf{v}}_1^T$ we find,

$$-2\overrightarrow{\mathbf{v}}_1^T A\overrightarrow{\mathbf{x}} + 2\overrightarrow{\mathbf{v}}_1^T \lambda_1 \overrightarrow{\mathbf{x}} + \lambda_2 \overrightarrow{\mathbf{v}}_1^T \overrightarrow{\mathbf{v}}_1 = 0 \tag{26}$$

$$-2\overrightarrow{\mathbf{v}}_1^T A\overrightarrow{\mathbf{x}} + 2\lambda_1 \overrightarrow{\mathbf{v}}_1^T \overrightarrow{\mathbf{x}} + \lambda_2 ||\overrightarrow{\mathbf{v}}_1|| = 0 \tag{27}$$

The middle term becomes zero as a result of our conditions. Additionally, $||\overrightarrow{\mathbf{v}}_1|| = 1$. So,

$$-2\overrightarrow{\mathbf{v}}_1^T A\overrightarrow{\mathbf{x}} + \lambda_2 = 0 \tag{28}$$

$$\lambda_2 = 2\overrightarrow{\mathbf{v}}_1^T A\overrightarrow{\mathbf{x}} \tag{29}$$

And by the properties of inner product,

$$\lambda_2 = 2\langle \overrightarrow{\mathbf{v}}_1, A\overrightarrow{\mathbf{x}} \rangle = 2\langle A\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{v}}_1 \rangle = 2\overrightarrow{\mathbf{x}}^T A^T \overrightarrow{\mathbf{v}}_1 = 2\overrightarrow{\mathbf{x}}^T A \overrightarrow{\mathbf{v}}_1 \tag{30}$$

And furthermore, from the previous part, $A\overrightarrow{\mathbf{v}}_1 = \mu_1 \overrightarrow{\mathbf{v}}_1$. Additionally, using $\langle \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{v}}_1 \rangle = 0$, we have,

$$\lambda_2 = 2\overrightarrow{\mathbf{x}}^T A\overrightarrow{\mathbf{v}}_1 = 2\mu_1 \overrightarrow{\mathbf{x}}^T \overrightarrow{\mathbf{v}}_1 = 0 \tag{31}$$

The Lagrangian then reduces to,

$$\frac{\mathcal{L}_{\lambda_1,\lambda_2}(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{v}}, \lambda_1, \lambda_2)}{\partial \overrightarrow{\mathbf{x}}} = -2A\overrightarrow{\mathbf{x}} + 2\lambda_1 \overrightarrow{\mathbf{x}} = 0 \tag{32}$$

And as before, we have,

$$A\overrightarrow{\mathbf{x}} = \lambda_1 \overrightarrow{\mathbf{x}} \tag{33}$$

Though this time, we label the eigenvalue, eigenvector pair as $(\mu_2, \overrightarrow{\mathbf{v}}_2)$.

(c) We now consider the optimization problem

$$\textbf{maximize} \quad \vec{\mathbf{x}}^T A \vec{\mathbf{x}} \quad \textbf{subject to} \quad ||\vec{\mathbf{x}}|| = 1 \textbf{ and } \langle \vec{\mathbf{x}}, \vec{\mathbf{v}}_1 \rangle = \langle \vec{\mathbf{x}}, \vec{\mathbf{v}}_2 \rangle = 0$$

Again, this problem admits a solution that we denote by $\vec{\mathbf{v}}_3$. Show that $\vec{\mathbf{v}}_3$ is an eigenvector of A that is orthogonal to $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

Again, we wish to minimize $-\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$ (which is equivalent to maximizing $\vec{\mathbf{x}}^T A \vec{\mathbf{x}}$) under the constraints that $||\vec{\mathbf{x}}|| = 1$, and $\langle \vec{\mathbf{x}}, \vec{\mathbf{v}}_1 \rangle = \langle \vec{\mathbf{x}}, \vec{\mathbf{v}}_2 \rangle = 0$. Then, by use of Lagrange multipliers, we have,

$$\begin{aligned} \mathcal{L}_{\lambda_i}(\vec{\mathbf{x}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \lambda_i) = -\vec{\mathbf{x}}^T A \vec{\mathbf{x}} + \lambda_1(\vec{\mathbf{x}}^T \vec{\mathbf{x}} - 1) + \lambda_2 \vec{\mathbf{x}}^T \vec{\mathbf{v}}_1 \\ + \lambda_3 \vec{\mathbf{x}}^T \vec{\mathbf{v}}_2 \end{aligned} \tag{34}$$

Then, we have,

$$\frac{\mathcal{L}_{\lambda_i}(\vec{\mathbf{x}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \lambda_i)}{\partial \vec{\mathbf{x}}} = -2A\vec{\mathbf{x}} + 2\lambda_1 \vec{\mathbf{x}} + \lambda_2 \vec{\mathbf{v}}_1 + \lambda_3 \vec{\mathbf{v}}_2 = 0 \tag{35}$$

If we multiply by $\vec{\mathbf{v}}_1^T$ we find,

$$-2\vec{\mathbf{v}}_1^T A \vec{\mathbf{x}} + 2\lambda_1 \vec{\mathbf{v}}_1^T \vec{\mathbf{x}} + \lambda_2 \vec{\mathbf{v}}_1^T \vec{\mathbf{v}}_1 + \lambda_3 \vec{\mathbf{v}}_1^T \vec{\mathbf{v}}_2 = 0 \tag{36}$$

The second and last terms become zero as a result of our conditions. Additionally, $||\vec{\mathbf{v}}_1|| = 1$, so,

$$-2\vec{\mathbf{v}}_1^T A \vec{\mathbf{x}} + \lambda_2 = 0 \tag{37}$$

Then we find the exact same problem encountered at the end of the previous problem, resulting in $\lambda_2 = 0$. Rewriting the Lagrangian yields,

$$\frac{\mathcal{L}_{\lambda_i}(\vec{\mathbf{x}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \lambda_i)}{\partial \vec{\mathbf{x}}} = -2A\vec{\mathbf{x}} + 2\lambda_1 \vec{\mathbf{x}} + \lambda_3 \vec{\mathbf{v}}_2 = 0 \tag{38}$$

Where now the problem is the same as the Lagrangian from the previous question. It is trivial to see that multiplying by $\overrightarrow{\mathbf{v}}_2^T$ would yield that $\lambda_3 = 0$. Then, finally, we are left with,

$$A\overrightarrow{\mathbf{x}} = \lambda_1\overrightarrow{\mathbf{x}} \tag{39}$$

Though this time, we label the eigenvalue, eigenvector pair as $(\mu_3, \overrightarrow{\mathbf{v}}_3)$.