DS-GA 1014 - Homework 2

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1. Which of the following are linear transformations? Justify.

For this problem, we will need to utilize the definition of linear transformation,

> *A linear transformation between two vector spaces V and W is a map $T : V \to W$ such that the following hold: (1) $T(\overrightarrow{\boldsymbol{v_1}} + \overrightarrow{\boldsymbol{v_2}}) = T(\overrightarrow{\boldsymbol{v_1}}) + T(\overrightarrow{\boldsymbol{v_2}})$ for any vectors $\overrightarrow{\boldsymbol{v_1}}, \overrightarrow{\boldsymbol{v_2}} \in V$ and (2) $T(c\overrightarrow{\boldsymbol{v}}) = cT(\overrightarrow{\boldsymbol{v}})$ for any $c \in \mathbb{R}$. [Def. 1]*

(a) T**:** $(x, y) \in \mathbb{R}^2 \to \ (x^2 + y^2, x - y) \in \mathbb{R}^2$

T is not a linear transformation because it fails conditions in *Definition 1.* Here, we show failure of condition (2),

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c^2(x^2 + y^2) \\ c(x - y) \end{bmatrix} \quad \neq \quad cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c(x^2 + y^2) \\ c(x - y) \end{bmatrix} \tag{1}$$

For some $c \in \mathbb{R}$.

(b) T**:** $(x, y) \in \mathbb{R}^2 \to \ (x + y + 1, x - y) \in \mathbb{R}^2$

T is not a linear transformation because it fails conditions in *Definition 1.* Here, we show failure of condition (2),

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c(x + y) + 1 \\ c(x - y) \end{bmatrix} \quad \neq \quad cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c(x + y + 1) \\ c(x - y) \end{bmatrix} \tag{2}$$

For some $c \in \mathbb{R}$.

(c) T**:** $A \in \mathbb{R}^{\mathrm{n\times m}} \to A^T \in \mathbb{R}^{\mathrm{m\times n}}$ **where** A^T **is transpose of** A**, i.e. the** $m \times n$ **matrix defined by** $(A^T)_{j,i} = A_{i,j}$ **for all** $(i,j) \in 1,...,m \times 1,...,n$.

T is a linear transformation because it satisfies both properties. First we show that it satisfies property (1):

For any two matrices, $A, B \in \mathbb{R}^{\mathrm{n\times m}}$, given by

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \qquad B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix} \tag{3}$$

We have that,

$$\begin{aligned}
(A+B)^T &= \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix} \right)^T \\
&= \left(\begin{bmatrix} (a+b)_{1,1} & \cdots & (a+b)_{1,m} \\ \vdots & \ddots & \vdots \\ (a+b)_{n,1} & \cdots & (a+b)_{n,m} \end{bmatrix} \right)^T \\
&= \begin{bmatrix} (a+b)_{1,1} & \cdots & (a+b)_{n,1} \\ \vdots & \ddots & \vdots \\ (a+b)_{1,m} & \cdots & (a+b)_{n,m} \end{bmatrix} \\
&= \begin{bmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{1,m} & \cdots & a_{n,m} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{n,1} \\ \vdots & \ddots & \vdots \\ b_{1,m} & \cdots & b_{n,m} \end{bmatrix} \\
&= A^T + B^T
\end{aligned} \tag{4}$$

This satisfies property (1). Now we show property (2). For any $c \in \mathbb{R}$,

$$\begin{aligned}
(cA)^T &= \left(c \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} \right)^T = \begin{bmatrix} ca_{1,1} & \cdots & ca_{1,m} \\ \vdots & \ddots & \vdots \\ ca_{n,1} & \cdots & ca_{n,m} \end{bmatrix}^T \\
&= \begin{bmatrix} ca_{1,1} & \cdots & ca_{n,1} \\ \vdots & \ddots & \vdots \\ ca_{1,m} & \cdots & ca_{n,m} \end{bmatrix} = c \begin{bmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{1,m} & \cdots & a_{n,m} \end{bmatrix} = c(A^T)
\end{aligned} \tag{5}$$

(d) T**:** $A \in \mathbb{R}^{\mathrm{n\times n}} \rightarrow$ $\mathbf{Tr}(A) \in \mathbb{R}$ **where** $\mathbf{Tr}(A)$ **is the trace of the matrix** A**, defined by,**

$$Tr(A) = \sum_{i=1}^{n} A_{i,i} \tag{6}$$

T is a linear transformation because it satisfies both properties. First we show that it satisfies property (1):

For any two matrices, $A, B \in \mathbb{R}^{\mathrm{n\times n}}$, we have,

$$\begin{aligned} Tr(A+B) &= \sum_{i=1}^{n}(A+B)_{i,i} = \sum_{i=1}^{n} A_{i,i} + B_{i,i} \\ &= \sum_{i=1}^{n} A_{i,i} + \sum_{i=1}^{n} B_{i,i} = Tr(A) + Tr(B) \end{aligned} \tag{7}$$

This satisfies property (1). Now we show property (2). For any $c \in \mathbb{R}$,

$$Tr(cA) = \sum_{i=1}^{n} cA_{i,i} = c\sum_{i=1}^{n} A_{i,i} = cTr(A) \tag{8}$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$f(1,2) = (1,2,3) \qquad \textbf{\textit{and}} \qquad f(2,2) = (1,0,1)$$

(a) Compute the matrix (canonically) associated to f.

Since the function f represents a transformation of some vector from $\mathbb{R}^2 \to \mathbb{R}^3$, we can model it as a matrix A such that $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{b}}$. So, we have,

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

Though matrix multiplication we arrive at three different systems of equations,

$$\begin{array}{c} a + 2b = 1 \\ 2a + 2b = 1 \end{array} \qquad \begin{array}{c} c + 2d = 2 \\ 2c + 2d = 0 \end{array} \qquad \begin{array}{c} e + 2f = 3 \\ 2e + 2f = 1 \end{array} \tag{10}$$

So, then we find $a = 0, b = 1/2, c = -2, d = 2, e = -2, f = 5/2$. So matrix A is given by,

$$\begin{bmatrix} 0 & 1/2 \\ -2 & 2 \\ -2 & 5/2 \end{bmatrix} \tag{11}$$

(b) Compute the set $\{x \in \mathbb{R}^2 | f(x) = (1,4,5)\}$.

We wish to solve the following,

$$\begin{bmatrix} 0 & 1/2 \\ -2 & 2 \\ -2 & 5/2 \end{bmatrix} \overrightarrow{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \tag{12}$$

Row reduction yields

$$\begin{bmatrix} -2 & 5/2 \\ -2 & 2 \\ 0 & 1/2 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \implies \begin{bmatrix} -2 & 5/2 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \tag{13}$$

Since every column has a pivot variable, we know that the dimension of the null space is zero. Therefore, there is only zero or one solutions. At this point we multiply, and find the general solution:

$$\begin{bmatrix} -2x_1 + \frac{5x_2}{2} \\ \frac{x_2}{2} \\ \frac{x_2}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \implies \vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{14}$$

So, the one and only solution is given above. The set of solutions is therefore $\{\vec{\mathbf{x}}\}$.

(c) Compute the set $\{x \in \mathbb{R}^2 | f(x) = (2, 4, 5)\}$

We wish to solve the following,

$$\begin{bmatrix} 0 & 1/2 \\ -2 & 2 \\ -2 & 5/2 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \tag{15}$$

Row reduction yields

$$\begin{bmatrix} -2 & 5/2 \\ -2 & 2 \\ 0 & 1/2 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \implies \begin{bmatrix} -2 & 5/2 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \tag{16}$$

Since every column has a pivot variable, we know that the dimension of the null space is zero. Therefore, there is only zero or one solutions. At this point we multiply, and find that no solution exists:

$$\begin{bmatrix} -2x_1 + \frac{5x_2}{2} \\ \frac{x_2}{2} \\ \frac{x_2}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \tag{17}$$

This implies that $x_2 = 4$ and $x_2 = 2$, which cannot be the case. Therefore the set of solutions is $\{\emptyset\}$.

3. Let $B \in \mathbb{R}^{4\times 3}$ be a matrix with arbitrary entries:

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,3} \end{bmatrix} \tag{18}$$

Find two matrices A and C such that

$$ABC = \begin{bmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} + B_{3,2} & B_{2,1} + B_{3,1} & B_{2,3} + B_{3,3} & B_{2,2} + B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{bmatrix} \tag{19}$$

Holds for any B defined above.

We conjecture that matrices A and C are the ones given below,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{20}$$

We can prove this through matrix multiplication. So,

$$ABC = A(BC) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} & B_{2,1} & B_{2,3} & B_{2,2} \\ B_{3,2} & B_{3,1} & B_{3,3} & B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{bmatrix} \tag{21}$$

And finally,

$$ABC = \begin{bmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} + B_{3,2} & B_{2,1} + B_{3,1} & B_{2,3} + B_{3,3} & B_{2,2} + B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{bmatrix} \tag{22}$$

4. (a) **Let A be a $n \times m$ matrix. Show that the image $Im(A)$ and the kernel $Ker(A)$ are subspaces of respectively $\mathbb{R}^n$ and $\mathbb{R}^m$.**

First, we show that $Ker(A)$ is a subspace of $\mathbb{R}^m$ if we are given some $n \times m$ matrix. Observe the following definition,

If A is some $n \times m$ matrix, then the null space of A, $\mathcal{N}(A)$, is defined by

$$\mathcal{N}(A) = \{\overrightarrow{\mathbf{x}} \in \mathbb{R}^m \,|\, A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}\} \tag{23}$$

[Def. 2]

Now, in order to show that $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^m$, we must show that (1) $\mathcal{N}(A)$ contains the zero-vector, (2) is closed under vector-addition, and is (3) closed under scalar multiplication.

We know that $\overrightarrow{\mathbf{0}} \in \mathcal{N}(A)$ since $\overrightarrow{\mathbf{0}} \in \mathbb{R}^m$ and $A\overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{0}}$. So condition (1) is satisfied.

Now, if we take some $\overrightarrow{\mathbf{x_1}} \in \mathcal{N}(A)$ and some $\overrightarrow{\mathbf{x_2}} \in \mathcal{N}(A)$ then it must follow that $\overrightarrow{\mathbf{x_1}} + \overrightarrow{\mathbf{x_2}} \in \mathcal{N}(A)$. Since $\overrightarrow{\mathbf{x_1}} \in \mathcal{N}(A)$, $A\overrightarrow{\mathbf{x_1}} = \overrightarrow{\mathbf{0}}$, and since $\overrightarrow{\mathbf{x_2}} \in \mathcal{N}(A)$, $A\overrightarrow{\mathbf{x_2}} = \overrightarrow{\mathbf{0}}$. So,

$$A\overrightarrow{\mathbf{x_1}} + A\overrightarrow{\mathbf{x_2}} = \overrightarrow{\mathbf{0}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{0}} \tag{24}$$

$$A(\overrightarrow{\mathbf{x_1}} + \overrightarrow{\mathbf{x_2}}) = \overrightarrow{\mathbf{0}} \tag{25}$$

Which implies that $\overrightarrow{\mathbf{x_1}} + \overrightarrow{\mathbf{x_2}} \in \mathcal{N}(A)$. This shows condition (2).

Finally, if we take some $\overrightarrow{\mathbf{x}} \in \mathcal{N}(A)$, it must follow that $c\overrightarrow{\mathbf{x}} \in \mathcal{N}(A)$ for any $c \in \mathbb{R}$. Since $\overrightarrow{\mathbf{x}} \in \mathcal{N}(A)$, $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$. So,

$$cA\overrightarrow{\mathbf{x}} = c\overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{0}} \tag{26}$$

$$A(c\overrightarrow{\mathbf{x}}) = \overrightarrow{\mathbf{0}} \tag{27}$$

Which implies that $c\overrightarrow{\mathbf{x}} \in \mathcal{N}(A)$. This shows condition (3). Therefore, $\mathcal{N}(A)$, synonymous with the $Ker(A)$, is a subspace of $\mathbb{R}^m$ since it is mapped out by the vectors $\{\overrightarrow{\mathbf{x}} \in \mathbb{R}^m \,|\, A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}\}$ and satisfies the conditions required by the definition of a subspace.

Now we show that the $Im(A)$ is a subspace of $\mathbb{R}^n$ if we are given some $n \times m$ matrix. Observe the following definition,

If A is some $n \times m$ matrix, then the image of A, $Im(A)$, is defined by

$$Im(A) = \{\overrightarrow{\mathbf{y}} \in \mathbb{R}^n \mid \overrightarrow{\mathbf{x}} \in \mathbb{R}^m : A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{y}}\} \tag{28}$$

[Def. 3]

Now, in order to show that $Im(A)$ is a subspace of $\mathbb{R}^n$, we must show that (1) $Im(A)$ contains the zero-vector, (2) is closed under vector-addition, and is (3) closed under scalar multiplication.

We know that $\overrightarrow{\mathbf{0}} \in Im(A)$ since this implies there is some $\overrightarrow{\mathbf{x}} \in \mathbb{R}^m$ such that $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$. This is always true if we take $\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$. So condition (1) is satisfied.

Now, if we take some $\overrightarrow{\mathbf{y_1}} \in Im(A)$ and some $\overrightarrow{\mathbf{y_2}} \in Im(A)$ then it must follow that $\overrightarrow{\mathbf{y_1}} + \overrightarrow{\mathbf{y_2}} \in Im(A)$. Since $\overrightarrow{\mathbf{y_1}} \in Im(A)$ and some $\overrightarrow{\mathbf{y_2}} \in Im(A)$, we know that there exists some $\overrightarrow{\mathbf{x_1}}$ and $\overrightarrow{\mathbf{x_2}}$ such that,

$$\begin{aligned} A\overrightarrow{\mathbf{x_1}} &= \overrightarrow{\mathbf{y_1}} \\ A\overrightarrow{\mathbf{x_2}} &= \overrightarrow{\mathbf{y_2}} \end{aligned} \tag{29}$$

Adding these equations and factoring implies,

$$\begin{aligned} A\overrightarrow{\mathbf{x_1}} + A\overrightarrow{\mathbf{x_2}} &= \overrightarrow{\mathbf{y_1}} + \overrightarrow{\mathbf{y_2}} \\ A(\overrightarrow{\mathbf{x_1}} + \overrightarrow{\mathbf{x_2}}) &= \overrightarrow{\mathbf{y_1}} + \overrightarrow{\mathbf{y_2}} \end{aligned} \tag{30}$$

This shows that $\overrightarrow{\mathbf{y_1}} + \overrightarrow{\mathbf{y_2}} \in Im(A)$ since there is some vector, namely $\overrightarrow{\mathbf{x_1}} + \overrightarrow{\mathbf{x_2}}$, which when transformed by the matrix A yields $\overrightarrow{\mathbf{y_1}} + \overrightarrow{\mathbf{y_2}} \in Im(A)$. So condition (2) is satisfied.

Finally, we show condition (3). If we take some $\overrightarrow{\mathbf{y}} \in Im(A)$, it must follow that $c\overrightarrow{\mathbf{y}} \in Im(A)$ for any $c \in \mathbb{R}$. Since $\overrightarrow{\mathbf{y}} \in Im(A)$, we know there exists some $\overrightarrow{\mathbf{x}}$ such that $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{y}}$. So, multiplying by c, we see,

$$\begin{aligned} A\overrightarrow{\mathbf{x}} &= \overrightarrow{\mathbf{y}} \\ cA\overrightarrow{\mathbf{x}} &= c\overrightarrow{\mathbf{y}} \\ A(c\overrightarrow{\mathbf{x}}) &= c\overrightarrow{\mathbf{y}} \end{aligned} \tag{31}$$

This shows that $c\overrightarrow{\mathbf{y}} \in Im(A)$ since there is some vector, namely $c\overrightarrow{\mathbf{x}}$, which when transformed by the matrix A yields $c\overrightarrow{\mathbf{y}} \in Im(A)$. So condition (3) is satisfied. Therefore, $Im(A)$ is a subspace of $\mathrm{I\!R}^{\mathrm{n}}$ since it is mapped out by the vectors $\{\overrightarrow{\mathbf{y}} \in \mathrm{I\!R}^{\mathrm{n}} \,|\, \overrightarrow{\mathbf{x}} \in \mathrm{I\!R}^{\mathrm{m}} : A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{y}}\}$ and satisfies the conditions required by the definition of a subspace.

(b) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{32}$$

Compute a basis of $Ker(A)$ and show that $Im(A) = \mathrm{I\!R}^3$.

We can find $Ker(A)$ by performing row reduction on A,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \implies A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \implies A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

From the last line of (24) we see that there are exactly three pivot columns and one free column. Thus, we now expect one vector to form our basis for $Ker(A)$, and it will result from $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$. We now have a system of equations, where we take the free variable $x_3 = t$. Then,

$$\begin{bmatrix} x_1 + 2x_2 + x_3 + 2x_4 \\ x_2 + x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \\ 0 \end{bmatrix} \tag{34}$$

Then the solutions to $A\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ are given by,

$$\vec{\mathbf{x}} = \begin{bmatrix} -t \\ 0 \\ t \\ 0 \end{bmatrix} \tag{35}$$

For any $t \in \mathbb{R}$. And a basis for $Ker(A)$ is then given by,

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{36}$$

Now we know two different properties of the rank, that allow us to show $Im(A) = \mathbb{R}^3$. Observe the first definition,

If A is some $n \times m$ matrix, then

$$rank(A) + dim(Ker(A)) = m \tag{37}$$

[Def. 4]

And furthermore, observe this second definition,

If A is some $n \times m$ matrix, then

$$rank(A) = dim(Im(A)) \tag{38}$$

[Def. 5]

So, with $dim(Ker(A)) = 1$ and $m = 4$, then by *Definition 4*, $rank(A) = 3$. By *Definition 5*, $dim(Im(A)) = 3$, which implies that $Im(A) = \mathbb{R}^3$.