DS-GA 1014 - Homework 3

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1. (2 points). Let $A \in \mathbb{R}^{n \times n}$.

(a) Show that if $A = \alpha Id_n$ for some $\alpha \in \mathbb{R}$, then for all $B \in \mathbb{R}^{n \times n}$ we have $AB = BA$.

Given that we have the definition,

If A is some $n \times n$ matrix, then

$$AId_n = Id_nA = A \tag{1}$$

[Def. 1]

Then, we can directly show the equality,

$$AB = BA \tag{2}$$
$$(\alpha Id_n)B = B(\alpha Id_n) \tag{3}$$
$$\alpha(Id_nB) = \alpha(BId_n) \tag{4}$$

And by *Definition 1*, we have, $Id_nB = BId_n = B$, so,

$$\alpha B = \alpha B \tag{5}$$

Thus we have shown that if if $A = \alpha Id_n$ for some $\alpha \in \mathbb{R}$, then for all $B \in \mathbb{R}^{n \times n}$ we have $AB = BA$.

(b) Conversely, show that if for all $B \in \mathbb{R}^{n \times n}$ we have $AB = BA$, then there exists $\alpha \in \mathbb{R}$ such that $A = \alpha Id_n$.

Given that $AB = BA$, we choose $B \in \mathbb{R}^{n \times n}$ such that $B = e_i e_j^T$. This makes B a matrix such that if B is formed of elements $b_{i,j}$ then B is 1 in the (i, j) element, and 0 everywhere else. Furthermore, we use the notation that A_i is the i-th column of the matrix A, and $A^{(j)}$ is the j-th row of matrix A. Then, we have,

$$AB = A(e_i e_j^T) = A_i e_j^T \tag{6}$$

$$BA = (e_i e_j^T)A = e_i A^{(j)} \tag{7}$$

$$A_i e_j^T = e_i A^{(j)} \tag{8}$$

The left side of this equation is a matrix with the i-th column of the matrix A, in its j-th column. The right side of the equation is a matrix, of the same size, with the j-th row of A in its i-th row. Comparing these two matrices, which must be equal, implies that all $a_{i,j} = 0$ unless $i = j$. Therefore, it is evident that only the diagonal of A are nonzero. Even more so, the elements of the diagonals are equal,

$$a_{i,i} = (AB)_{i,j} = (BA)_{i,j} = a_{j,j} \tag{9}$$

Since the diagonal of A is completely populated, and all entries are equal, we have that,

$$A = \alpha Id_n \tag{10}$$

2. (3 points). Let $M \in \mathbb{R}^{n \times m}$ and $r = rank(M)$. Show that there exists $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{r \times m}$ such that $M = AB$.

Given some matrix $M \in \mathbb{R}^{n \times m}$ such that $r = rank(M)$, a rank of r implies the columns of M, denoted as $\overrightarrow{\mathbf{m}_1}, ..., \overrightarrow{\mathbf{m}_m}$ are in the span of some r vectors in $\mathbb{R}^n$. In other words, $\overrightarrow{\mathbf{m}_1}, ..., \overrightarrow{\mathbf{m}_m} \in Span(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_r})$ for some $\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_r} \in \mathbb{R}^n$. We then set the columns of A to be the vectors $\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_r}$, which yields the desired shape of A, $A \in \mathbb{R}^{n \times r}$. So far, we have

$$\left[\overrightarrow{\mathbf{m}_1} \middle| \; ... \; \middle| \overrightarrow{\mathbf{m}_m} \right] = \left[\overrightarrow{\mathbf{v}_1} \middle| \; ... \; \middle| \overrightarrow{\mathbf{v}_r} \right] \left[\overrightarrow{\mathbf{b}_1} \middle| \; ... \; \middle| \overrightarrow{\mathbf{b}_m} \right] \tag{11}$$

Where we have set $b_1, ..., b_m$ to be the columns of B such that $b_1, ..., b_m \in \mathbb{R}^r$. We hope to show,

$$M = AB \tag{12}$$

As given in (11) as well. We know that for any $i \in \{1, ..., m\}$, we have,

$$\left[\overrightarrow{\mathbf{m}_i} \right] = \left[\overrightarrow{\mathbf{v}_1} \middle| \; ... \; \middle| \overrightarrow{\mathbf{v}_r} \right] \left[\overrightarrow{\mathbf{b}_i} \right] = b_{i_1} \left[\overrightarrow{\mathbf{v}_1} \right] + ... + b_{i_r} \left[\overrightarrow{\mathbf{v}_r} \right] \tag{13}$$

Where b_{i_j} is the j-th component of vector $\overrightarrow{\mathbf{b}_i}$. Since, for any $\overrightarrow{\mathbf{m}_i}$, $\overrightarrow{\mathbf{m}_i} \in Span(\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_r})$, we see that we can take the correct linear combination of vectors $\overrightarrow{\mathbf{v}_1}, ..., \overrightarrow{\mathbf{v}_r}$ to yield m_i - we do this by selecting the values of b_{i_j}. Since this can be done for all $i \in \{1, ..., m\}$, we have that there exists $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{r \times m}$ such that $M = AB$ if $r = rank(M)$.

3. (3 points). Let $A \in \mathbb{R}^{n \times m}$.

(a) Let $M \in \mathbb{R}^{m \times m}$ be an invertible matrix. Show that

$$rank(AM) = rank(A) \tag{14}$$

Since we have the following proposition,

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$, then $rank(AB) \leq rank(A)$. [Prop. 1]

Therefore, we know that,

$$rank(AM) \leq rank(A) \tag{15}$$

But since M is invertible, we also have that,

$$rank((AM)M^{-1}) \leq rank(AM) \tag{16}$$

By *Proposition 1*. But,

$$rank((AM)M^{-1}) = rank(AId) = rank(A) \tag{17}$$

Putting this all together yields the expression,

$$rank(A) = rank((AM)M^{-1}) \leq rank(AM) \leq rank(A) \tag{18}$$

This forces the equality of the expression we set out to prove. Since M is an invertible matrix, we have,

$$rank(AM) = rank(A) \tag{19}$$

(b) Let $M \in \mathbb{R}^{n \times n}$ be an invertible matrix. Show that

$$rank(MA) = rank(A) \tag{20}$$

In order to show the above equality, we first set out to prove that $Ker(MA) = Ker(A)$. In order to show this, we first show that $Ker(A) \subset Ker(MA)$.

Observing the definition of the null space of A, or $Ker(A)$, we have,

> *If A is some $n \times m$ matrix, then the null space of A, $\mathcal{N}(A)$, is defined by*
>
> $$\mathcal{N}(A) = \{\vec{\mathbf{x}} \in \mathbb{R}^m \,|\, A\vec{\mathbf{x}} = \vec{\mathbf{0}}\} \tag{21}$$
>
> *[Def. 2]*

So, for any $\vec{\mathbf{x}} \in Ker(A)$, it follows that,

$$\begin{aligned} A\vec{\mathbf{x}} &= \vec{\mathbf{0}} \\ MA\vec{\mathbf{x}} &= M\vec{\mathbf{0}} = \vec{\mathbf{0}} \end{aligned} \tag{22}$$

So, therefore, $\vec{\mathbf{x}} \in Ker(MA)$. This suffices to show that $Ker(A) \subset Ker(MA)$. Now we show that $Ker(MA) \subset Ker(A)$.

Proof by contradiction. Suppose that M is an invertible, singular matrix and $Ker(MA) \not\subset Ker(A)$. Then this implies that there exists some vector $\vec{\mathbf{x}}$ such that $MA\vec{\mathbf{x}} = \vec{\mathbf{0}}$ but $A\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$. Call $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$. Then this implies that $M(\vec{\mathbf{y}}) = \vec{\mathbf{0}}$ with $\vec{\mathbf{y}} \neq \vec{\mathbf{0}}$. From the following theorem,

> *If $M \in \mathbb{R}^{n \times n}$ is an invertible matrix, then $Ker(M) = \vec{\boldsymbol{0}}$. [Thm. 1]*

We know that it cannot be the case that $M(\vec{\mathbf{y}}) = \vec{\mathbf{0}}$ with $\vec{\mathbf{y}} \neq \vec{\mathbf{0}}$, since $Ker(M)$ contains only the zero vector. So we have show by contradiction that $Ker(MA) \subset Ker(A)$.

Since we have show that $Ker(A) \subset Ker(MA)$ and $Ker(MA) \subset Ker(A)$, it follows that $Ker(MA) = Ker(A)$. Now, we also have that,

> *If A is some $n \times m$ matrix, then*

$$rank(A) + dim(Ker(A)) = m \tag{23}$$

[Thm. 2]

So it follows that,

$$rank(A) + dim(Ker(A)) = m \tag{24}$$

$$rank(MA) + dim(Ker(MA)) = m \tag{25}$$

And again, since $Ker(MA) = Ker(A)$, we have proven that,

$$rank(MA) = rank(A) \tag{26}$$

4. (2 points). The trace $Tr(M)$ of a $k \times k$ matrix M is defined as the sum of its diagonal coefficients, i.e.

$$Tr(M) = \sum_{i=1}^{k} M_{i,i} \tag{27}$$

(a) Let $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ and $B \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$. Show that $Tr(AB) = Tr(BA)$.

If we have two matrices, $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ and $B \in \mathbb{R}^{\mathrm{m}\times\mathrm{n}}$, the shape of AB is $n \times n$ and the shape of BA is $m \times m$, then we have,

$$(AB)_{i,i} = \sum_{k=1}^{m} a_{i,k} b_{k,i} \tag{28}$$

$$(BA)_{k,k} = \sum_{i=1}^{n} b_{k,i} a_{i,k} \tag{29}$$

So then it follows that,

$$\begin{aligned} Tr(AB) &= \sum_{i=1}^{n} (AB)_{i,i} = \sum_{i=1}^{n} \sum_{k=1}^{m} a_{i,k} b_{k,i} \\ &= \sum_{k=1}^{m} \sum_{i=1}^{n} b_{k,i} a_{i,k} = \sum_{k=1}^{m} (BA)_{k,k} \\ &= Tr(BA) \end{aligned} \tag{30}$$

(b) For $A, B, C \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, do we have $Tr(ABC) = Tr(CAB) = Tr(ACB)$?

It is false that $Tr(ABC) = Tr(CAB) = Tr(ACB)$. Observe the following example for $A, B, C \in \mathbb{R}^{3\times3}$, where the trace is calculated in Python,

```
import numpy as np
A = np.array([[1,2,1],[4,2,5],[2,5,6]])
B = np.array([[1,0,1],[4,-2,5],[2,5,-1]])
C = np.array([[1,1,1],[2,2,3],[-2,5,4]])

def tdot(a,b,c):
    t = np.dot(b,c)
    f = np.dot(a,t)
    return(f)
print('Tr(ABC): ' + str(np.trace(tdot(A,B,C))))
print('Tr(CAB): '+ str(np.trace(tdot(C,A,B))))
print('Tr(ACB): '+ str(np.trace(tdot(A,C,B))))
```

```
Tr(ABC): 280
Tr(CAB): 280
Tr(ACB): 318
```

In this example, it is evident that $Tr(ABC) = Tr(CAB) = Tr(ACB)$ does not hold since $280 = 280 \neq 318$.